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ON THE EXACT MINIMUM VARIANCE HEDGE OF AN UNCERTAIN QUANTITY WITH FLEXIBILITY

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Summary: The purpose of this paper is to investigate the impact of production cost variability upon hedging decision when the firm is a risk minimizer agent facing both price and quantity uncertainties. We show, under a perfect flexibility assumption, that considering cost variability leads to a lower [higher] optimal hedge ratio assuming a positive [negative] relation between prices and quantities.

Keywords and phrases: Minimum variance hedge, uncertain demand, perfect flexibility, cost function.

JEL Classification Numbers: D21, D81, G11.

1 Introduction

Since the pioneering contributions by Rotschild and Stiglitz (1970), Baron (1970) and Sandmo (1971), the theory of the competitive firm under price uncertainty has been a focus of much attention from financial literature. In the so-called 'unbiased' case, Danthine (1978), Holthausen (1979) and Feder et al. (1980) studied consequences of the introduction of forward markets by establishing the now well-known *separation property*[1]. Relaxing the non-realistic assumption of unbiasedness when markets are organized and standardized – futures markets – Ederington (1979) elicited the optimal minimum variance hedge ratio[2]. Taking into account the inescapable basis risk[3], this ratio is still today the most widely used, because of limited improvements provided by numerous suggested alternatives[4].

However, these alternatives, as the original Ederington paper, all consider a fixed amount of output[5]. In other words, the quantity to hedge is perfectly known before the hedging decision is made. Naturally, economic situations that have not this property are frequent. It is not difficult to find examples of production posterior to the hedging decision: (*i*) farmers never know perfectly the volume of their future crop, because they necessarily depend on meteorology and other factors; (*ii*) power producers and petroleum companies face uncertainty in quantity because of non-expected variations in demand; (*iii*) multinational firms do not know exactly the amount they will perceive in foreign currencies

[1] Production decisions are not affected by differences in risk aversion or in price expectations. However, hedging decisions depend on both risk preferences and price expectations. See also Ethier (1973).

[2] Ederington showed the relationship existing between optimal hedging and the futures prices/spot prices covariance.

[3] Basis risk occurs because of location, timing or quality differences between production and futures contracts specifications.

[4] Among others: expected utility ratio, mean-variance ratio, semivariance ratio, Sharpe's ratio, mean-Gini coefficient. For a survey, see Chen et al. (2003). For a critique see Moosa (2003).

[5] We can precise here that output problem and input problem are symmetric under assumption of input inflexibility (see Anderson & Danthine (1983), p 379).

in advance. On this subject, few contributions can be mentioned.

First, McKinnon (1967) in an agricultural framework shows the importance of the covariance between quantity and price for variance minimization . The problem is that McKinnon does not benefit from Ederington's work and then does not put forward covariance between spot and futures prices. Losq (1982) generalizes McKinnon's model in an expected utility framework. Preferences are then not necessarily quadratic and joint probability distribution not necessarily normal. But Losq's analysis does not assume any production cost and the model is built exactly as if the decision-maker only consider its income. To some extent, the analysis of Kerkvliet and Moffett (1991) is near enough because of the no production cost assumption. Authors consider the case of a multinational firm, which will receive a uncertain amount of foreign currency in the future. The firm is assumed to be risk-averse and plans a risk-minimizing hedge. The optimal hedge ratio is derived and shown to be depending on the covariance between prices and quantities. Lapan and Moschini (1994) provide a general model in an expected-utility framework. They assume a production cost, but correspondingly to the agricultural reality, the cost is proportional to the crop area. Effectively in agricultural domain no real adjustment can be realized once the area is decided.

This paper considers the case of a firm facing both price and output uncertainties, but whose production perfectly matches demand. An example is the case of a power producer. In that case supply exactly corresponds to consumption and none kWh is produced without demand. Thus variability on demand – quantity – leads to variability on production cost, and cost cannot therefore been looked as a fixed amount. Accordingly, previous analysis are not relevant. The aim of this paper is to show the difference between previous optimal hedge ratios (OHR) and optimal ratio with perfect flexibility. We then indicate that without taking into account flexibility, the OHR is systematically biased, following the intuition that effectiveness of the hedge is depending on the statistical relationship between the hedge instrument and the product to be hedge.

The paper is set out as follows. Section 2 presents the model with the introduction of a variable cost function. Section 3 gives analytical results. Section 4 summarises the main conclusions of the paper.

2 The model

The model is a two periods model. Consider a competitive firm with a given - deterministic - production technology, which produces a certain commodity. Its production capacity is chosen prior to the model.Output is produced at a cost $C(q)$, increasing, but indifferently concave or convex[6]. The cost function is assumed deterministic. The firm is assumed to face a stochastic spot price $\tilde{p_1}$ for its single output in the second period $(t = 1)$[7].

In addition, the firm faces a quantity uncertainty in that the demand $\tilde{q}$ is not known in the first period $(t = 0)$. Because of its flexibility property, the firm can perfectly match the demand level. In this way the issue differs fundamentally from the standard newsboy problem examined throughout operational research literature and initiated by [14].

The only decision variable for the firm is the amount of output hedged h in the futures market[8]. The current futures price f_0 is perfectly known, whereas the second period's one $\tilde{f_1}$ is not. The realized total profit is then:

$$\widetilde{\Pi} = \tilde{p_1}\tilde{q} - C(\tilde{q}) + h(\tilde{f_1} - f_0) \tag{1}$$

Consider the firm as infinitely risk-averse[9], its aim is to minimize the profit's variance

[6]See discussion in the next section.

[7]Throughout the paper, random variables have a tilde.

[8]Futures contract is the only type of hedging instrument or insurance available to the firm.

[9]Risk aversion is a realistic assumption, but note that even risk neutral agents can exhibit apparent risk aversion. For instance, because of market imperfections (see Greenwald & Stiglitz (1993)), liquidity

without taking into account consequences of the hedge on expectation of profit:

$$\min_h [var(\widetilde{\Pi})] \tag{2}$$

Taking into account variability of production cost, profit variance is:

$$var[\widetilde{\Pi}] = var[\tilde{p_1}\tilde{q}] + var[C(\tilde{q})] + h^2 var[\tilde{f_1}] - 2cov[\tilde{p_1}\tilde{q}, C(\tilde{q})] + 2hcov[\tilde{p_1}\tilde{q}, \tilde{f_1}] - 2hcov[C(\tilde{q}), \tilde{f_1}] \tag{3}$$

An expression of the variance of a product of random variables can be found in (Bohrnstedt and Golberger, p 1439, equation (5)). However, this result is not essential because the firm has no power to reduce this variance by hedging[10]. From a certain viewpoint, this term can be seen as an irreducible risk, a risk on which the firm has no control.

Consider that the only one manner the firm can reduce its profit's variance is hedging. Consider further that only one futures contract is available. The first order condition (henceforth FOC) for program (2) is[11]:

$$h^* var[\tilde{f_1}] + cov[\tilde{p_1}\tilde{q}, \tilde{f_1}] - cov[C(\tilde{q}), \tilde{f_1}] = 0 \tag{4}$$

A simplification of equation (4) is essential to make the hedge ratio usable. Let us consider $C(\tilde{q})$ as a random variable. For any pair of random variables x and y, $cov(x, y) = E(xy) - E(x)E(y)$. We can then rewrite $cov[C(\tilde{q}), \tilde{f_1}]$ as the difference between $E[C(\tilde{q})\tilde{f_1}]$ and $E[C(\tilde{q})]E[\tilde{f_1}]$. To still reduce the result, a preliminary proposition is useful.

Lemma 1 (Price's Theorem, 1958) [12] *Let x and y be bivariate normally distributed with covariance σ_{xy}. Then if $h(x)$ and $g(x)$ are two functions square integrable with*

constraints (see Vercammen (1994)) or taxation convexity (see Smith & Stulz (1985)).

[10]There is no relation between the variance of the revenue and h, the number of futures contracts.

[11]The second-order condition is satisfied given positivity of a variance.

[12]For a similar result in a more general framework, see also Middleton (1948).

respect to the normal density and with derivatives of all orders,

$$E[h(x)g(y)] = E[h(x)]E[g(y)] + \sigma_{xy}E[h'(x)]E[g'(y)] +$$

$$\sigma_{xy}^2 E[h''(x)]E[g''(y)] + ... + \sigma_{xy}^i E[h^{(i)}(x)]E[g^{(i)}(y)] + ...$$

In particular, $E[xg(y)] = E[x]E[g(y)] + \sigma_{xy}E[g'(y)]$

Price's theorem allows to write:

$$E[C(\tilde{q})\tilde{f}_1] = E[C(\tilde{q})]E[\tilde{f}_1] + cov(\tilde{f}_1, \tilde{q})E[C'(\tilde{q})] \tag{5}$$

Using this last result, the product of expectations vanishes in last expression and:

$$cov[C(\tilde{q}), \tilde{f}_1] = E[C'(\tilde{q})]cov(\tilde{f}_1, \tilde{q}) \tag{6}$$

To still simplify equation (4), another preliminary result is useful[13].

Lemma 2 (Bohrnstedt and Goldberger, 1969) *Let x, y and z be jointly distributed random variables, then (with $cov(.,.)$ and $E(.)$ respectively covariance and expectation operator):*

$$cov(xy, z) = E(x)cov(y, z) + E(y)cov(x, z) + E[(x - E(x))(y - E(y))(z - E(z))]$$

Further, under multivariate normality, all third moments vanish. We have $E[(x - E(x))(y - E(y))(z - E(z))] = 0$ and last equation is reduced to: $cov(xy, z) = E(x)cov(y, z) + E(y)cov(x, z)$

We can then rewrite the second quantity in left-hand side of equation (4)

$$cov(\tilde{p_1}\tilde{q}, \tilde{f}_1) = E(\tilde{q})cov(\tilde{p_1}, \tilde{f}_1) + E(\tilde{p_1})cov(\tilde{q}, \tilde{f}_1) + E[(\tilde{q} - E(\tilde{q}))(\tilde{p_1} - E(\tilde{p_1}))(\tilde{f}_1 - E(\tilde{f}_1))] \tag{7}$$

[13]This result is commonly used in this kind of problems with multiple sources of uncertainty, as soon as one of the risks applies in a multiplicative manner. See for instance Lapan and Moschini (1994) or Kerkvliet and Moffett (1991).

Using Bohrnstedt and Goldberger's hypothesis concerning multivariate normality, (7) becomes:

$$cov(\tilde{p_1}\tilde{q}, \tilde{f_1}) = E(\tilde{q})cov(\tilde{p_1}, \tilde{f_1}) + E(\tilde{p_1})cov(\tilde{q}, \tilde{f_1}) \tag{8}$$

By integrating (6) and (8) in condition (4), we obtain:

$$h^* var[\tilde{f_1}] + [E(\tilde{q})cov(\tilde{p_1}, \tilde{f_1}) + E(\tilde{p_1})cov(\tilde{q}, \tilde{f_1})] - [E[C'(\tilde{q})]cov(\tilde{f_1}, \tilde{q})] = 0 \tag{9}$$

Analysis of condition (9) allows to determine the optimal hedge ratio which minimizes the profit's variance.

3 Optimal hedge with perfect flexibility

As mentioned in introduction, flexibility can often be observed in economics, especially in network activities. For lots of these activities, production exactly matches demand. Hence, the variable cost is also exactly corresponding to the quantity supplied, as soon as cost function is deterministic. This is the case for instance in power production where the variable cost is quasi perfectly equal to the raw material consumed in order to produce.

Proposition 1 *Exact variance minimizing optimal hedge ratio with perfect flexibility is given by:*

$$h^* = \frac{E[C'(\tilde{q})]cov(\tilde{f_1}, \tilde{q}) - E(\tilde{p_1})cov(\tilde{f_1}, \tilde{q}) - E(\tilde{q})cov(\tilde{p_1}, \tilde{f_1})}{var(\tilde{f_1})} \tag{10}$$

Corollary 1 *If prices and quantities are positively correlated, the optimal hedge ratio is lower if cost variability is taken into account.*

Influence of the new element on the OHR is depending on the sign of $cov(\tilde{f_1}, \tilde{q})$ because $E[C'(\tilde{q})]$ is always positive. There is a difference with the agricultural approach here.

Following McKinnon, "*any particular farmer expects his own output to be positively correlated with the aggregate output of all farmers and hence negatively correlated with prices*". This assumption appears particularly relevant when the uncertainty considered and meteorology are linked. In our case, the opposite may occurs. Power markets are today often managed by auctions. A high-level demand logically leads to higher profits for electricity producers, because of the unique price system. A positive relation between individual output and prices can therefore be expected. As a consequence, the hedge ratio is moderated if we assume that the hedger position is short in futures contracts. An immediate conclusion is that when production cost variability is not taken into account, the hedge ratio is systematically overvaluated. The difference between the initial ratio and the ratio proposed here varies according to the marginal cost value in the area of uncertainty. There is an intuition here. In an area of high marginal cost, the hedge is statistically less adapted – in probabilities – and the the hedge ratio is then lower compared to a low marginal cost area, where a variation in quantity has a lower impact on the variation of production cost.

To appreciate our general solution, we can precise that in special cases, previous results provided in the literature can be recognized. Firstly, if production are completely ignored, the result is similar to Kerkvliet & Moffett (equation 16). Secondly, if the firm is infinitely risk-averse the solution is then identical to the Lapan & Moschini ratio (equation 34 and equation 41 for the mean-variance extension). Finally, the original result from Ederington is derived by assuming a non random quantity.

4 Conclusion

Absence of flexibility means that effective cost of production is determined *ex ante*, without any dependence *vis-à-vis* the realized demand. In our paper, the production cost is now assumed to be random - through quantity uncertainty - and optimal hedge ratio can be derived. Previous articles gave a random characteristic to quantity by using random production function (Just and Pope, 1979) or an uncertain amount of money. Here, uncertainty comes from demand. In this case, residual cost variability must unambiguously be taken into account, particularly if marginal costs are high.

References

[1] Anderson R.W., Danthine J.-P.: Hedger diversity in futures markets, Economic Journal **93**, 370-89 (1983)

[2] Baron D.P.: Price Uncertainty, utility and industry equilibrium in pure competition, International Economic Review **11**, 463-80 (1970)

[3] Bohrnstedt G.W., Goldberger A.S.: On the exact covariance of products of random variables, Journal of American Statistical Association **64**, 1439-42 (1969)

[4] Chen S.-S., Lee C.-F., Shrestha K.: Futures hedge ratios: a review, Quarterly Review of Economics and Finance **43**, 433-65 (2003)

[5] Danthine J.-P.: Information, futures market and stabilizing speculation, Journal of Economic Theory **17**, 79-98 (1978)

[6] Ederington L.H.: The hedging performance of new futures markets, Journal of Finance **34**, 157-70 (1979)

[7] Ethier W.: International trade and the forward exchange market, American Economic Review **63**, 494-503 (1973)

[8] Feder G., Just R.E., Schmitz A.: Futures markets and the theory of the firm under price uncertainty, Quarterly Journal of Economics **95**, 317-28 (1980)

[9] Greenwald B.C., Stiglitz J.E.: Financial market imperfections and business cycles, Quarterly Journal of Economics **108**, 77-114 (1993)

[10] Holthausen D.M.: Hedging and the competitive firm under price uncertainty, American Economic Review **69**, 989-95 (1979)

[11] Just R.E., Pope R.D.: Stochastic specification of production functions and economic implications, Journal of Econometrics **7**, 67-86 (1979)

[12] Kerkvliet J., Moffett M.H.: The hedging performance of new futures markets, Journal of Financial and Quantitative Analysis **68**, 944-50 (1991)

[13] Lapan H., Moschini G.: Futures hedging under price, basis, and production risk, American Journal of Agricultural Economics **76**, 465-477 (1994)

[14] Leland H.E.: Theory of the firm facing uncertain demand, American Economic Review, **62**, 278-91 (1972)

[15] Losq E.: Hedging with price and output uncertainty, Economics Letters **10**, 65-70 (1982)

[16] McKinnon R.I.: Futures markets, buffer stocks, and income stability for primary producers, Journal of Political Economy **75**, 844-61 (1967)

[17] Middleton D.: Some general results in the theoty of noise through nonlinear devices, Quarterly Applied Mathematics **5**, 445 (1948)

[18] Moosa I.: The sensisivity of optimal hedge ratio to model specification, Finance Letters **1**, 15-20 (2003)

[19] Price R.: A useful theorem for nonlinear devices having gaussian inputs, I.R.E. Transport PGIT, IT **4**, 69 (1958)

[20] Rotschild M., Stiglitz J.E.: Increasing risk: I. A definition, Journal of Economic Theory **2**, 225-43 (1970)

[21] Sandmo A.: On the theory of the competitive firm under price uncertainty, American Economic Review **61**, 65-73 (1971)

[22] Smith C.W., Stulz R.: The determinants of firms' hedging policies, Journal of Financial and Quantitative Analysis **20**, 391-405 (1985)

[23] Vercammen J.: Capital constraints and optimal hedging in factor markets, Economics Letters **45**, 433-439 (1994)

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