CROSS HEDGING and LIQUIDITY:
A NOTE

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CROSS HEDGING AND LIQUIDITY: A NOTE

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Abstract: Cross hedging is a way to improve statistical hedge results because of markets’ incompletion. In this framework, several markets instead of just one market, are used to increase the hedger’s financial possibilities. In the Anderson-Danthine model (1981), the optimal hedge in the multivariate case is described and commented, but transaction costs are neglected. The aim of this note is to suggest a new version of the initial model, in which transaction costs are now taken into account. In a first step, benchmark case is formalized with deterministic costs. Secondly, we consider stochastic liquidity and statistical links between liquidity levels. In the first case, the intuitive non-optimality is shown as soon as transaction costs are integrated. In the second case, a more general model is suggested and a link is mentioned with the ”commonality in liquidity” concept.

Key Words: cross hedging, liquidity, mean-variance utility, commonality in liquidity, transaction costs, hedging

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1 Introduction

The problem of transaction costs is very often neglected in the hedging literature. In fact, references on this issue, old (Telser (1955), Miller (1965), Demsetz (1968), Telser and Hinginbotham (1977), Cohen, Maier, Schwartz and Whitcomb (1981)) or more recent (Bessembinder and Kaufman (1997), Locke and Venkatesh (1997), Jouini, Koehl and Touzi (1997), Wang, Yau and Baptiste (1997), Tse and Zabotina (2001), Domowitz, Glen and Madhavan (2001)) discuss generally about quality or costs only, but no link is made with optimal hedge.

In the Cross Hedging Model from Anderson and Danthine (1981), there is evidently no consideration about the transaction cost problem. As we can observe in their fourth proposition:

"(4) In the absence of transactions costs and of a "perfect hedge", hedgers will take advantage of the better risk-return performance associated with simultaneous trades in a multiplicity of futures contracts”, p 1183.

Since this seminal work, lots of theoretical and empirical studies have arisen. For instance agricultural economic studies, exchange rate studies, financial assets studies or energy studies showed the possibilities of cross hedging. Among these studies, only Brorsen, Buck and Koontz (1998) allude to implicit transaction costs, those linked to market liquidity.

The aim of this last study is to show that when a choice is possible between two trading places for near products, hedgers can be conducted to switch as soon as the liquidity differential is too high. The methodology consist in estimating the liquidity on the two stock exchanges considered and integrating these estimations in the profit function of the hedger. However results are not particularly convincing in this case, because of the low difference between the two liquidity levels. In this note, we will show that this methodology can be used in the multi-products case, and therefore modifying the optimal vector of hedging.

This result is, to our sense, fundamental for four reasons:

1. In a perspective of dynamic hedging, where hedge position is at least modified every day and often several times a day, transaction costs have an increasing place in the financial budget of the agent.
2. The liberal wave and the following deregulation have modified and increased the hedging needs of firms (see Tussing and Hatcher (1994)) because of the risky involved (see Wilson (2002)).

3. The high degree of concurrence existing in these financial markets, explained by global information, and arbitrage possibilities.

4. The lack in the hedging theory about transaction costs. As mentioned by Lien and Tse (2002) in their survey, liquidity issue is not taken into account in no method.

Therefore this paper will integrate implicit transaction costs in the general Anderson-Danthine model to show that the optimal hedge vector is modified. In a first step we will adopt a deterministic approach - not far from Brorsen and al. (1995), but in the multivariate cas. Secondly, a stochastic approach will be retained. The aim is here to link portfolio choice with ”commonality in liquidity” concept from Chordia, Roll and Subrahmanyam (2001).

Anderson-Danthine’s model and optimal hedge vector

Anderson-Danthine’s model has been elaborated during the post perfect-hedge period (symbolized for instance by Hieronymus (1971)). Previous contributions advised to take on the futures market an exactly (same absolute value) opposed position than in cash market. Financial models derived from portfolio theory (Johnson (1960) and Stein (1961)) do not confirm this paradigm, but they are concentrated on cash and futures simultaneously decision. Ederington (1979), Holthausen (1979) and Figlewski (1984) will give an extension to Working’s work on basis risk (1953), and demonstrate that there is no reason for futures position to have the same absolute value (but in the opposite sense) than cash position. More than confirming the no-full-hedge paradigm, Anderson-Danthine will also use a multi-market framework and illustrate the speculative term of the optimal hedge.

The model is a two-period model \( t = 0, 1 \) with only one risky asset, whom prices are respectively \( p_0 \) et \( p_1 \) in the first and second period. Quantity \( y \) owned of the risky asset is considered as a constant during the two periods. We assume \( n \) futures markets. Positions for each futures contract represent the column-vector \( f \). Prices of futures contracts are \( p_0^f \) et \( p_1^f \). Authors do not take into account the necessity for the number of contracts purchased to be integers. Without loss of generality, we also do not take into account the production cost, and then the expected profit can be written as follows:

\[
\tilde{\Pi} = \tilde{p}_1 y - (\tilde{p}_1^f - p_0^f) y^f
\]

\[\text{Authors explained in 1994 that energy financial markets have no interest in a monopoly situation. Future, with deregulation, will confirm the increasing interest of firms for hedging.} \]

\[\text{Following the author, absence of efficient futures markets could explain, in a part, California collapse.} \]

\[\text{Full-hedge or routine-hedge are also used.} \]

\[\text{See also Rolfo (1980) and Paroush and Wolf (1986).} \]

\[\text{Tilde always mention random variables.} \]
We assume a risk-averse decision-maker with a mean-variance utility function (see Eichner and Wagener for the link between mean-variance utility and general expected utility functions).

**Remark 1** Benninga, Eldor and Zilcha (1983, 1984) showed that mean-variance analysis can be reduced to a variance minimization analysis in the unbiasedness case (conditions are discussed in Cecchetti, Cumby and Figlewski (1988) or Lence (1995)). In this case, there is no benefit opportunity for exchanging futures contracts because of the perfect equality between spot and futures prices at the end of the hedge operation. But this case is an exception, occurring when hedge finishes exactly at the futures contract date (see Bryis, Crouhy et Schlesinger (1993)).

However, in the multi-product case there is absolutely no reason to assume unbiasedness, as mentioned by Brorsen and al. (1998), p 450): ”All hedges are cross hedges to some degree because of differences in grades, location, and time”. Therefore a mean-variance utility function will represent decision-maker preferences:

\[ U(\tilde{\Pi}) = E(\tilde{\Pi}) - \frac{1}{2} \alpha V(\tilde{\Pi}) \] (2)

with \( \alpha \) risk-aversion coefficient in the Arrow-Pratt sense and profit as mentioned in (Hedge optimization problem conduct then to maximize utility with \( f \) as the multidimensional decision variable, or:

\[ \max_f [E(\tilde{\Pi}) - \frac{1}{2} \alpha V(\tilde{\Pi})] \] (3)

\( \Sigma \) is the \( n+1 \) variance-covariance matrix of futures prices and spot price. We can write:

\[ \Sigma = \begin{pmatrix} \sigma_{00} & \Sigma_{01} \\ \Sigma_{10} & \Sigma_{11} \end{pmatrix} \]

with \( \text{var}(\tilde{p}_1) = \sigma_{00}, \Sigma_{10} \), the column vector of covariances between spot price and futures prices, \( \Sigma_{01} \), the transposed and \( \Sigma_{11} \), the variance-covariance futures prices matrix.

The first order condition (FOC)\(^{12}\) for (4)

\[ \frac{\partial E(\tilde{\Pi})}{\partial f} = (p_0 f - \tilde{p}_1 f) - \alpha(\Sigma_{11}.f - y.\Sigma_{10}) = 0 \] (4)

If \( \Sigma_{11} \) is not singular, we can calculate the unique optimal hedge vector for a fixed production \( y \):

\(^{12}\)Second order condition is satisfied because of the concavity of the utility function.
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\[ f = \frac{1}{\alpha} \Sigma^{-1}_{11} (p_0^f - \bar{p}_1^f) + y \Sigma^{-1}_{11} \Sigma_{10} \]  

(5)

This first step is not a global optimization process, but a second-best decision. In fact, production is fixed, and integrated as an exogenous variable in the model. However, this process is not marginal in economics, where hedging decisions are often taken in a second step, because of the anteriority of production decisions.

The result (1)

1. \( \frac{1}{\alpha} \Sigma^{-1}_{11} (p_0^f - \bar{p}_1^f) \), or the pure speculative component (corresponding to the \( y = 0 \) case)
2. \( y \Sigma^{-1}_{11} \Sigma_{10} \), or the pure hedge component, varying with the production value

The pure speculative component is a classical optimal fund as described in the portfolio choice theory. This component is a part, inversely related to the risk-aversion coefficient, of the optimal fund (constructed with all necessary futures contracts). There is no need to own physical goods to desire the optimal fund and then this fund is also desired by speculators. The pure hedge component corresponds to a minimization-variance part, as proposed for instance by Ederington (1979), but in the multivariate case.

The original Anderson-Danthine model gives then optimal hedging decision when there is no forward contract available. For authors, it is then logical to consider several futures contracts because: "However, many cash goods do not have obvious futures. In these cases a "cross-hedge" may attempted by taking position in a futures for a related commodity" (p 1187). The considered cost is the sum of implicit costs (liquidity and microstructure characteristics) and explicit costs (taxes, commissions...).

2 Deterministic transaction costs

The question is whether optimal hedge vector is modified when transaction costs are integrated in the analysis? In this section, we consider transaction costs as exogenous. Their value is known for each futures contract. We also consider a unique transaction cost at the first period instead of two costs for two periods.

Liquidity issue is perhaps the most broad topic of modern finance. More than a financial markets characteristic, liquidity is simply the essence of markets, their reason to exist. For Orléan (1999), liquidity is not an intrinsical property of the asset, as the fundamental value. Liquidity is a human invention, product of the organized markets concept. Further its nature, liquidity measure is also a discussable issue. A general way to evaluate liquidity is to

\footnote{Note that a rigorous analysis should consider two different costs, the first for the entry [resp., the exit], the second for the exit [resp., the entry]. But without loss of generality, we can consider just a unique average cost.}

\footnote{Le pouvoir de la finance, p 32.}
estimate the bid-ask spread for the asset. Half of this spread represents (implicit) transaction costs. A presentation of the principal models of bid-ask spread evaluation\textsuperscript{15} can be found in O’Hara (1995)\textsuperscript{16} and Biais, Foucault and Hillion (1997)\textsuperscript{17}.

Let $TC_{\text{spot}}$ be the total transaction cost on the spot market and $TC_i^f$ be the transaction costs on the $n$ futures markets. Profit can be rewritten:

\[
\tilde{\Pi}_{TC\text{det}} = (\tilde{p}_1 - TC_{\text{spot}})y - (\tilde{p}_1^f - p_0^f - TC^f)f
\]  

With an identical utility function than in the original model, FOC-vector and optimal hedge vector follow from (\textsuperscript{16})

\[
\frac{\partial E(\tilde{\Pi}_{TC\text{det}})}{\partial f} = (p_0^f + TC_{\text{spot}} - \tilde{p}_1^f) - \alpha(\Sigma_{11}f - y\Sigma_{10}) = 0
\]  

\[
f_{TC\text{det}} = \frac{1}{\alpha}\Sigma^{-1}_{11}(p_0^f + TC_{\text{spot}} - \tilde{p}_1^f) + y\Sigma^{-1}_{11}\Sigma_{10}
\]  

This result, if not very different from the original one, at a first sight, leads to several remarks which constitute a first proposition.

\textbf{Proposition 1} \textit{In the case of deterministic transaction costs:}

1. Transaction costs on spot market have no influence on the pure speculative part of the optimal hedge.
2. Optimal speculative fund is modified and become $\Sigma^{-1}_{11}(p_0^f + TC_{\text{spot}} - \tilde{p}_1^f)$.
3. Optimal hedge vector is modified due to the $\Sigma^{-1}_{11}$ matrix.

Beyond the intuition of non-optimality when transaction costs are not taken into account, we can now observe the transformation mechanism of the optimal hedge vector by the $\Sigma^{-1}_{11}$ matrix. The first item of the proposition explains that spot market transaction costs have just an influence on the \textit{ex ante} production decision. In commodities markets, the possibility of differences between liquidity levels are non-negligible. Often, near commodities are quoted in several markets, and some of them are obviously more liquid than others.

This first result is a confirmation of the empirical work due to Brorsen and al. (1998). It confirms that part of the hedge can be transferred in another exchange place because of transaction costs differential. Let us examine consequences of liquidity considered henceforth as a random variable.

\textsuperscript{15}We can cite Roll (1984), Glosten (1987), Stoll (1989), George, Kaul and Nimalendran (1993), and the general model of Roomans (1993), which is a generalization of its predecessors.

\textsuperscript{16}Market Microstructure Theory.

\textsuperscript{17}Microstructure des marchés financiers.
3 Stochastic transaction costs

In the last section, we show that hedging behavior is modified when a liquidity constant term is included in the analysis. But we are not nearer of the truth. Recent econometric studies showed that a time-varying formalization for liquidity gave more satisfying results (see for instance Engle and Lange (1997)\textsuperscript{18} or Watanabe (2003)\textsuperscript{19}). Our aim here, is not to formalize liquidity but just to examine consequences of liquidity randomization.

Let now transaction costs, as described in last section, be $\tilde{T}C^f$\textsuperscript{20}. Profit can then be written:

$$\tilde{\Pi}_{TC,sto} = \tilde{p}_1 y - (\tilde{p}_1^f - p_0^f - \tilde{T}C^f)' f$$

(9)

Remark 2 This time we do not take into account neither production costs nor transaction costs on spot market. In a dynamic hedging perspective, these costs are negligible compared to futures markets transaction costs. In addition, our interest in this paper is not on relation between futures position and physical or cash position.

Expected profit (\( E(\tilde{\Pi}_{TC,sto}) \))

$$E(\tilde{\Pi}_{TC,sto}) = \tilde{p}_1 y - (\tilde{p}_1^f - p_0^f - \tilde{T}C^f)' f$$

(10)

$$V(\tilde{\Pi}_{TC,sto}) = V(\tilde{p}_1 y) + V[p_0^f + \tilde{T}C^f - \tilde{p}_1^f]' f + 2cov(\tilde{p}_1 y, p_0^f + \tilde{T}C^f - \tilde{p}_1^f)' f$$

(11)

We now introduce \((n+1, n+1)\) variance/covariance transaction costs matrix, constructed on the same model than the \(\Sigma\) matrix of the original model:

$$TC = \begin{pmatrix} \sigma_{00} & TC_{01} \\ TC_{10} & TC_{11} \end{pmatrix}$$

with \(var(\tilde{p}_1) = \sigma_{00}\),

\(TC_{10}\), the \(n\) column vector of covariances between spot price and transaction cost on each futures market,

\(TC_{01}\), its transposed, and

\(TC_{11}\), the \((n, n)\) variance/covariance transaction costs matrix.


\textsuperscript{19}Watanabe M., A Model of Stochastic Liquidity, Yale Working Paper no.03-18.

\textsuperscript{20}A \(n\) column-vector.
We also construct the \((n, n)\) covariance matrix, representing statistical relation between futures prices and transaction costs:

\[
Q = \text{cov}(\tilde{T}C_i^f, \tilde{p}_j^f)
\]

We can then write the FOC on \(f\):

\[
\frac{\partial E(\tilde{\Pi}_{TCsto})}{\partial f} = [p_0^f + \tilde{T}C_f^f - \tilde{p}_1^f] - \alpha[(TC_{11} + \Sigma_{11} - Q).f + y.(TC_{10} - \Sigma_{10})] = 0
\]  (12)

and the following optimal hedge vector:

\[
f_{TCsto} = \frac{1}{\alpha}Z^{-1}[p_0^f + \tilde{T}C_f^f - \tilde{p}_1^f] + yZ^{-1}(TC_{10} - \Sigma_{10})
\]  (13)

with \(Z\) the \((n, n)\) matrix defined by: \(Z = TC_{11} + \Sigma_{11} - Q\) and supposed non-singular.

**Proposition 2** With stochastic transaction costs, behavior which do no take into account co-variance between different liquidity levels is quasi-surely non-optimal. The optimal hedge vector is given by \(f_{TCsto} = \frac{1}{\alpha}Z^{-1}[p_0^f + \tilde{T}C_f^f - \tilde{p}_1^f] + yZ^{-1}(TC_{10} - \Sigma_{10})\).

This proposition refers to the well-known diversification concept from Markowitz (1952). By this way, it is shown that the optimal hedge vector is not modified uniquely by transaction cost values, but also by statistical relation between this values. Each asset is in fact considered as the sum of two random variables, whom relationships must be integrated in the analysis.

4 Concluding remarks and extensions

Since Markowitz (1952) and his diversification principle, neglect covariances is not more than an aberration. In addition than demonstrating that transaction costs have a real effect on optimal hedging, we show that randomization brings even more complexities in the analysis. Very often, transaction costs are a really minor part of total costs, but in particular cases, as for instance in emerging markets, they are not. On these markets, liquidity differentials are remarkable and then logical arbitrages must be done. Cross-hedging is a manner to do arbitrage trading.

A relevant concept in financial literature for our analysis is the ”commonality in liquidity” due to Chordia, Roll and Subrahmanyam (2001). The authors establish the concept of market liquidity, comparable with the market return of the CAPM framework. Proper liquidity is then evaluated for each asset, and compared to market liquidity, exactly as the well-known coefficient \(\beta\) of Sharpe (1964). A link with this theory could represent one particular interesting extension to this note.
References


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